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BOEING SCIENTIFIC RESEARCH LABORATORIES

Minimizing Functionals on Normed-Linear Spaces

A. A. Goldstein

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ERRATA AND ADDITIONS

"Minimizing Functionals on Normed-Linear Spaces"

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Page 3. In (b) of the theorem replace $L_p[0,1]$ by a uniformly convex Banach space.

Page 4. Line 7 from bottom. Assume E is uniformly convex. By [6] p. 113.

Page 8. Line 6. for $p > 1$.

Page 10. Insert after sentence of line 5: and f is uniformly F-differentiable on S if $p > 2$. The inequality $|a|^{2r} + |b|^{2r} -2|a|^r|b|^r[a,b]/|a||b| \leq |a|^{2r-2}(r^2 + 1)|a - b|^2$ where $|a| > |b|$, and a direct computation show that the F-derivative f' exists and is lipschitz continuous for all $p > 1$.

Page 11. Line 12. Delete paragraph beginning with "By the theorem of I..." Replace this with the following: Moreover, the space $\mathcal{L}_r$ is uniformly convex. This follows by a theorem of Smulian [12], which states that if the norm in a B-space is uniformly F-differentiable on the unit sphere, then the conjugate space is uniformly convex.

Page 13. Replace ref [6] by M. Day, Normed Linear Spaces, *Academic Press*, N. Y., 1962.

Add:

[12] V. Smulian. Sur la derivabilité de la normed dans l'espace de Banach, C. R. (Doklady), *Acad. Sci. U.R.S.S.*, Vol. 27, (1940), p. 643-648.

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MINIMIZING FUNCTIONALS ON NORMED-LINEAR SPACES

by

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ABSTRACT

This paper extends results of [1], [2], of Goldstein, and [3] of Vainberg concerning steepest descent and related topics. An example is given taken from a simple rendezvous problem in control theory. The problem is one of minimizing a norm on an affine subspace. The problem here is solved in the primal. A solution in the dual is given by Neustadt [4].

I. GENERATION OF MINIMIZING SEQUENCES

Let E be a normed linear space (n. l. space), x_0 an arbitrary point of E and f a functional defined on E. Let S denote the level set $\{x \in E : f(x) \leq f(x_0)\}$ defined at an arbitrary fixed $x_0 \in E$. We denote by $f'(x)$ the Frechet or F-derivative of f at x. We call f *uniformly F-differentiable* on S if f is F-differentiable on S and if $\delta(\varepsilon)$ in the definition of the F-derivative is constant on S. The F-derivative of f at x will be denoted by $f'(x)$. If $g \in E^*$ the value of g at x will be denoted by $[g,x]$, and if $h \in E^{**}$ the value of h at $g \in E^*$, by $[h,g]$. Recall that if E and F are n. l. spaces and A is bounded linear operator from E to F, in short $A \in B(E,F)$, and if A is onto then A^{-1} exists and belongs to $B(F,E)$ if and only if for some $m > 0$ and all $x \in E$, $||Ax|| \geq m||x||$; and that $m||x|| \leq ||Ax|| \leq M||x||$ for all x in E implies that $M^{-1}||y|| \leq ||A^{-1}y|| \leq m^{-1}||y||$ for all y in F.

We observe that if E is a reflexive Banach Space, $A \in B(E,E^*)$ and $[Ax,x] \geq m||x||^2$ for all $x \in E$, then A is onto and thus has an inverse. For, on the contrary supposition, take $f^0 \notin M = \text{range } A$. Choose g in E^{**} such that $g(f^0) = \text{dist}(f^0,M) > 0$, $||g|| = 1$ and $g(f) = 0$ for all f in M. Take $\bar{g}$ in E so that $[g,f] = [f,\bar{g}]$ for all f in E^*. Then $0 = [g,Ax] = [Ax,\bar{g}]$ for all x in E.

Thus $[A\bar{g},\bar{g}] = 0$ while $||g|| = ||\bar{g}|| = 1$. Q. E. D.

Let ϕ denote a bounded map from S to E satisfying the two conditions $[f'(x),\phi(x)] \geq 0$, and given $\varepsilon > 0$ there exists $\delta > 0$ such that $[f'(x),\phi(x)] < \delta$ implies $||f'(x)|| < \varepsilon$. Some examples of such mappings are the following:

(1) Let $A \in B(E^*,E)$ such that $[y,Ay] \geq \sigma||y||^2$ for all $y \in E^*$ and some $\sigma > 0$. Let $\phi(x) = Af'(x)$ and choose $\delta = \varepsilon^2\sigma$. Then $||f'(x)|| < \varepsilon$.

As a possible candidate for the operator A, suppose f is twice F-differentiable on E. Assume that for some $\mu > 0$, and some x in S the operator $f''(x)$ in $B(E,E^*)$ is onto and is *"bounded below"*, that is, the bilinear functional satisfies $[f''(x)z,z] \geq \mu||z||^2$ for all z in E. Then $||f''(x)z|| \geq \mu||z||$ showing that $f''(x)$ has an inverse $[f''(x)]^{-1} = A \in B(E^*,E)$. Since A has a bounded inverse, there exists a number $\sigma > 0$ such that $||Ay|| \geq \sigma||y||$ for all $y \in E^*$. Set $z = Ay$. Then $[f''(x)z,z] = [y,Ay] \geq \mu\sigma^2||y||^2$ showing the candidacy of A.

(2) Suppose E is a reflexive Banach space. By the weak compactness of the unit sphere in E it follows that for some z_0, $||z_0|| = 1$, $[f'(x),z_0] = ||f'(x)||$. Set $\phi(x) = z_0||f'(x)||$. Because $[f'(x),\phi(x)] = ||f'(x)||^2$, $\phi(x)$ is the analogue of the gradient in Hilbert space. When E is an L_p space the point z_0 is obtained by considerations of equality in Hölder's inequality.

(3) Since $||f'(x)|| = \sup\{[f'(x),z]:||z|| = 1\}$, if $0 < \alpha < 1$ a point z_0 exists such $[f'(x),z_0] \geq \alpha||f'(x)||$. If for fixed α and all $x \in S$ we can find such z_0, we may take $\phi(x) = z_0||f'(x)||$.

In what follows let $\Delta(x,\rho) = f(x) - f(x - \rho\phi(x))$ and $g(x,\rho) = \Delta(x,\rho)/\rho[f'(x),\phi(x)]$. Assume E is a normed-linear space and S is the level set of f at x_0 in E. In what follows, assume $0 < \sigma < \frac{1}{2}$.

Theorem. Assume that on S f is uniformly F-differentiable or that the F-derivative f' exists and is uniformly continuous. Set $x_{k+1} = x_k$, when $[f'(x_k),\phi(x_k)] = 0$; otherwise choose* ρ_k so that $\sigma < g(x_k,\rho_k) \leq 1-\sigma$ when $g(x_k,1) < \sigma$ or $\rho_k = 1$ when $g(x_k,1) \geq \sigma$, and set $x_{k+1} = x_k - \rho_k\phi(x_k)$.

(a) If S is bounded or f is bounded below then $\{f'(x_k)\}$ converges to 0 while $\{f(x_k)\}$ converges downward to a limit, L. If S is compact, then every cluster point of $\{x_k\}$ is a zero of f'. In addition, if $\phi(x_k) \to 0$ and f' has finitely many zeros, $\{x_k\}$ converges.

(b) If S is convex and bounded and f is convex, $L = \inf\{f(x):x \in S\} = \theta$. If, in addition, E is a reflexive Banach space, then every weak cluster point of $\{x_k\}$ minimizes f on E. If $E = L_p[0,1]$, then $\{x_k\}$ converges to a unique minimizer of f.

(c) Assume that the Gateaux derivative f'' exists on S and satisfies $\mu||z||^2 \leq [f''(x)z,z] \leq M||z||^2$ for all $x \in S$, $z \in E$ and some $\mu > 0$. Assume S is convex and E is complete. Then $\{x_k\}$ converges to a unique minimizer of f on E.

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*If the Gateaux differential f'' satisfies $f''(x,h,h) \leq ||h||^2/\rho_0$ for all h in E, x in S and some $\rho_0 > 0$ we can choose ρ_k to satisfy $\delta \leq \rho_k \leq 2\rho_0 - \delta$ with $0 < \delta \leq \rho_0$. The method of steepest descent could also be employed. See [9].

The proof of (a) is given in [1]. The proof there is stated for E, a Hilbert space, but the same proof works when E is taken to be a n. l. space. Two comments might be made, however. S bounded and f' uniformly continuous on S implies that f' is bounded on S. (See e.g. [5], p. 19.) It follows by employing the mean value theorem that f is bounded below on S. The statements f uniformly F-differentiable and the F-derivative f' is uniformly continuous are equivalent. (See [5], p. 45.)

(b) Given $\varepsilon > 0$ choose $z' \in E$ such that $f(z') = \theta + \varepsilon/2$. Because f' exists at x_k and f is convex, $f(z') \geq f(x_k) + [f'(x_k), z' - x_k]$. Since $\{f'(x_k)\} \to 0$ and S is bounded, for all k sufficiently large $f(x_k) \leq f(z') + \varepsilon/2 = \theta + \varepsilon$, showing that $L = \theta$.

If E is reflexive and S is convex, closed, and bounded, then S is weakly compact. Since f is convex, the sets $\{x \in E : f(x) \leq k\}$ are closed, convex, and weakly closed, for all k. Thus f is weakly lower semi-continuous. If z is a weak cluster point of $\{x_k\}$ then for an appropriate subsequence, $\liminf f(x_k) = L \geq f(z)$. Assume $E = L_p[0,1]$ and f is the norm on E. By [6], p. 78, if $\{x_k\}$ converges weakly to z and $f(x_k) \to z$ then $\{x_k\}$ converges strongly to z. It follows that every weak cluster point of $\{x_k\}$ is a strong cluster point of $\{x_k\}$. Since f' vanishes at every weak cluster point of $\{x_k\}$ and f' vanishes only once by the strict convexity of f, every subsequence of $\{x_k\}$ has the same weak cluster point z, showing that $\{x_k\}$ converges to z.

(c) The hypothesis of (c) imply that f' is Lipschitz continuous and that the set S is bounded. Otherwise S would contain an unbounded sequence, say $\{z_k$
By Taylor's theorem if $u \varepsilon S$, $f(z_k) \geq f(u) + ||z_k - u||[(||z_k - u||\mu/2 - ||f'(u)||]$, showing that $f(z_k) \geq f(x_0)$, for large k, whence S must be bounded.
We now show that the sequence $\{x_k\}$ is Cauchy. Again by Taylor's theorem if $s > k$, $f(x_s) - f(x_k) \geq [f'(x_k), x_s - x_k] + \mu||x_s - x_k||^2/2$. Since S is bounded, $||x_s - x_k|| \leq D$ where D is the diameter of S. Thus
$||x_s - x_k||^2 \leq \frac{2}{\mu}\{f(x_s) - f(x_k) + D||f'(x_k)||\}$ which shows that $\{x_s\}$ is a Cauchy sequence. By the completeness of E $\{x_s\}$ has a limit, say z, in E, and $f'(z) = 0$. If z is not unique, then $f'(z_1) = f'(z_2) = 0$, $z_1 \neq z$. Thus $f(z_1) - f(z_2) \geq \frac{\mu}{2}||z_1 - z_2||^2 \leq f(z_2) - f(z_1)$, a contradiction.
Hence z is unique and is a minimizer of f.

Remarks: Useful remarks may be found in [1], [3] and [9].

II. NEWTONIAN STEPS AND ACCELERATION

Suppose that at the given point x_0, the function f satisfies the conditions of the first example, namely $\phi(x) = f''_{-1}(x_0)f'(x)$, where

$f''_{-1}(x_0) = [f''(x_0)]^{-1}$. The corresponding iteration is $x_{n+1} = x_n - \rho_n f''_{-1}(x_0)f'(x_n)$.
This algorithm, when $\rho_n \equiv 1$ is known as the "modified" Newton's method (see [3], p. 259, or [7], p. 696). In a similar manner if $f''_{-1}(x)$ exists and is uniformly bounded below on S, we may define $\phi(x_n) = f''_{-1}(x_n)f'(x_n)$. We shall do this below. It is clear from what has already been said that ϕ satisfies hypotheses of the above theorem. Our object now is to formulate

an algorithm using $f''_{-1}(x_n)f'(x_n) = \phi(x_n)$ which will converge at a superliner rate.

In the following we set $\Delta(x,\rho) = f(x) - f(x - \rho f''_{-1}(x)f'(x))$ and $g(x,\rho) = \Delta(x,\rho)/\rho[f''_{-1}(x)f'(x),\ f'(x)]$.

Theorem. Assume the level set S is a convex subset of a Banach space E. For each x in S assume the F-derivative f'' is continuous on S, $f''(x)$ is onto, $||f''(x)|| \leq M$, and $[f''(x)z,z] \geq m||z||^2$ for some $m > 0$ and all z in E. Set $x_{k+1} = x_k - \rho_k f''_{-1}(x_k)f'(x_k)$, where ρ_k is chosen so that for $\theta < 1/2$, $0 < \theta \leq g(x_k,\rho_k) \leq 1 - \theta$ with $\rho_k = 1$ if possible. Then:

(a) There exists a number N such that if $k > N$ then $\rho_k = 1$.

(b) There is a unique minimizer of f and the sequence $\{x_k\}$ converges to it faster than any geometric progression.

Proof. We have for all x in S that $M||z||^2 \geq [f''(x)x,x] \geq m||z||^2$ and $m^{-1}||y||^2 \geq [y,f''_{-1}(x)y] \geq mM^{-2}||y||^2$. Thus if $\phi(x) = f''_{-1}(x)f'(x)$, then $[f'(x),\phi(x)] \geq mM^{-2}||f'(x)||^2$, showing that ϕ satisfies the conditions of the above theorem. Since f'' is bounded on S, f' is Lipschitz continuous, by the mean value theorem. By (c) above $\{x_k\}$ converges to a unique minimizer of f.

Expand $\Delta(x,\rho)$ to two terms in the Taylor series with remainder $[f''(\xi)h,h)$, where $h = \rho f''_{-1}(x)f'(x)$. Set $f''(\xi) = f''(x) + f''(\xi) - f''(x)$. Ther $g(x,\rho) = 1 - \rho/2 - \rho[(f''(\xi) - f''(x))f''_{-1}(x)f'(x), f''_{-1}(x)f'(x)]/2[f'(x),f''_{-1}(x)f'(x)]$ $\geq 1 - \rho/2 - \rho||f''(\xi) - f''(x)||M^2/2m^3$. Thus $|g(x,\rho) - 1 + \rho/2| \leq \rho||f''(\xi) - f''(x)||M^2/2m$. Since $\xi(\rho_k)$ lies between x_k and x_{k+1} $x_0,\xi_0,x_1,\xi,\ldots$

is a Cauchy sequence; and it, together with its limit z, is a compactum. Consequently, on this compactum f'' is uniformly continuous, so that $\{||f''(\xi(\rho_k) - f''(x)||\}$ converges to 0, showing that the choice $\rho_k = 1$ is eventually feasible.

To prove (b) we write $x_{k+1} - z = x_k - z - \rho_k f''_{-1}(x_k)f'(x_k) =$ $x_k - z - \rho_k f''_{-1}(x_k)f''(x_k)(x_k - z) + \rho_k f''_{-1}(x_k)[f''(x_k)(x_k - z) - f'(x_k)]$.

Thus $||x_{k+1}-z|| = ||x_k-z-\rho_k(x_k-z)|| + \rho_k||f''_{-1}(x_k)||\,||f''(x_k)(x_k-z)-f'(x_k)||$.

Since f' is F-differentiable at x_k, $||f'(z) - f'(x_k) - f''(x_k)(z - x_k)|| < \varepsilon\,||z - x_k||$. Thus $||x_{k+1} - z|| = (1 - \rho_k)||x_k - z|| + \rho_k m^{-1}\varepsilon||z - x_k||$ Q.E.D.

Remarks:

(1) Both sides of the inverse of $f''_{-1}(x)$ are used in the proof.

(2) The analogue of the modified Newton process, namely choosing $\phi(x) = f''_{-1}(x_0)f'(x)$ or $f''_{-1}(x_k)f'(x)$ with k fixed also will under the hypothesis of the above theorem generate a sequence converging to a unique minimizer of f. Since $||x_{k+1}-z|| = ||x_k-z-\rho_k f''_{-1}(x_0)f''(z)(x_k-z)|| + \rho_k||[f''_{-1}(x_0)]||\,\varepsilon\,||x_k-z||m^{-1}$ when $||x_k-z|| < \delta$, the rate of convergence is eventually geometric provided $||I - \rho_k f''_{-1}(x_0)f''(z)|| < 1$. Since

$||I - \rho_k f''_{-1}(x_0)f''(z)|| \le 1 - \rho_k + \rho_k||f''_{-1}(x_0)||\ ||(f''(x_0) - f''(z)||$, if $||f''(x_0) - f''(z)||$ is sufficiently small, $\rho_k \equiv 1$ will generate a sequence converging to z at the rate of geometric progression. A sufficient condition for the global geometric convergence would be $(M/m) < 1/2$, since $||f''(x)|| \le M$ and $||f''_{-1}(x)|| \le m^{-1}$.

(3) Pertinent remarks may be found in [3].

III. EXAMPLE

a) We consider the following problem which arises from a linearized rendezvous problem. See for example [8], [9] and [4]. In [4] this problem is solved in the "dual". We consider here a construction in the "primal". In [8] and [9], we have discussed this problem in the spaces $\mathcal{L}_1$ and $\mathcal{L}_2$; we now discuss the problem in $\mathcal{L}_p$ for $p > 2$. Let $\mathcal{L}_p$ denote the direct sum of n $L_p[0,1]$ spaces. Thus a point $x \in \mathcal{L}_p$ if $x = (x_1,\ldots,x_n)$ and $x_i \in L_p[0,1]$; the norm in $\mathcal{L}_p$ will be $||x||_p = [\int_0^1 |x(t)|^p dt]^{1/p}$ where $|x(t)| = \left[\sum_{i=1}^{n} x_i^2(t)\right]^{\frac{1}{2}}$. Since $\sqrt{n}\max\{x_i(t) : 1 \leq i \leq n\} \geq |x(t)|$, $||x||_p$ is well defined. Let $\{u^i : 1 \leq i \leq m\}$ be a linearly independent set in $\mathcal{L}_p$. Set $\frac{1}{p} + \frac{1}{q} = 1$. Since $q < p$, u^i is also in $\mathcal{L}_q$. Given numbers α_i, $(1 \leq i \leq m)$ define the affine subspace $M = \{x \in \mathcal{L}_p : [u^i,x] = \alpha_i : 1 \leq i \leq m\}$. We shall consider the problem of minimizing $f(x) = ||x||_p^p$ on M. The limits $p \to 1$ and $p \to \infty$ correspond to the cases of rendezvous with minimum fuel and minimum thrust amplitude respectively. In what follows we shall assume for simplicity that $n = 2$. There are no further difficulties in the general case.

We first observe that if the Gateaux differential (G-differential) of f exists it is given by:

$$f'(x)h = p\int_0^1 |x(t)|^{p-2}[x_1(t)h_1(t) + x_2(t)h_2(t)]dt$$

$$= p\int_0^1 |x(t)|^{p-1}\left[\frac{x_1(t)}{|x(t)|}h_1(t) + \frac{x_2(t)}{|x(t)|}h_2(t)\right]dt$$

$$\leq p||x||_p^{p/q}[||h_1||_p + ||h_2||_p].$$

Here Holder's inequality has been employed on the function $t \to |x(t)|^{p-1}$ which belongs to $L_q[0,1]$. We have also used $||\cdot||_p$ for the norm in $L_p[0,1]$. Thus the G-derivative of f exists.

Observe now that if the second G-differential exists it is given by:

$$[f''(x)h,k] = p(p-2)\int_0^1 |x(t)|^{p-4}(x_1(t)h_1(t) + x_2(t)h_2(t))(x_1(t)k_1(t) + x_2(t)k_2(t))dt$$

$$+ p\int_0^1 |x(t)|^{p-2}(k_1(t)h_1(t) + k_2(t)h_2(t))dt$$

$$= p(p-2)\int_0^1 |x(t)|^{p-2}\left(\frac{x_2(t)}{|x(t)|}h_1(t) + \frac{x_2(t)}{|x(t)|}h_2(t)\right)\left(\frac{x_1(t)}{|x(t)|}k_1(t) + \frac{x_2(t)}{|x(t)|}k_2(t)\right)dt$$

$$+ p\int_0^1 |x(t)|^{p-2}(k_1(t)h_1(t) + k_2(t)h_2(t))dt$$

$$\leq 2p(p-2)\int_0^1 |x(t)|^{p-2}|h(t)||k(t)|dt + p\int_0^1 |x(t)|^{p-2}|h(t)||k(t)|dt.$$

If $x \in L_p$, $y \in L_q$ and $z \in L_r$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$, then $\int_0^1 |x(t)y(t)z(t)|dt \leq ||x||_p||y||_q||z||_r$. Since the function $t \to |x(t)|^{p-2}$ belongs to $L_{p'}$, where $p' = \frac{p}{p-2}$, and $\frac{1}{p'} + \frac{2}{p} = 1$ it follows that if $u(t) = |x(t)|^{p-2}$, $\left[\int_0^1 |u(t)|^{p'}dt\right]^{1/p'} = ||x||_p^{p-2}$, and $[f''(x)h,k] \leq (2p^2 - 3p)||x||_p^{p-2}||h||_p||k||_p$.

As before, let S denote the level set of f at x_0 where x_0 will be

subsequently chosen in M. Thus if $x \in S$, $||x||_p^{p-2} = [f(x)]^{\frac{p-2}{2}} \leq [f(x_0)]^{\frac{p-2}{2}}$ showing that $[f''(x)h,k]$ is uniformly bounded on S, if h and k are confined to the unit sphere. It follows by Taylor's theorem that f' is F-differentiable on S. By the generalized mean value theorem it further follows that f' is Lipschitz continuous on S.

We now construct x_0 on M. Let $x_0 = \Sigma c_j u^j$. Thus x_0 lies on M if and only if $\Sigma_j c_j [u^i, u^j] = \alpha_i$. We show that the null space of the matrix $\{[u^i, u^j]\}$ consists only of the 0 element so that c_i is uniquely determined. If for some $c_j \neq 0$, $\Sigma_j c_j [u^i, u^j] = 0$ then $\Sigma_i c_i \Sigma_j c_j [u^i, u^j] = [\Sigma_i c_i u^i, \Sigma_j c_j u^j] = 0$, contradicting the linear independence of the set $\{u^i : 1 \leq i \leq n\}$. Let $N = \{x \in \mathcal{L}_p : [u^i, x] = 0, i=1,\ldots,m\}$. We now choose h to maximize $[f'(x),h]$ subject to $||h||_p = 1$ and $h \in N$. The maximum is achieved because the sphere meets N in a weakly compact set and the linear function $[f'(x),\cdot]$ is weakly continuous. The maximization can be accomplished by the method of Euler multipliers [10]. Let $\phi(h) = ||h||_p^p - 1$ and $\phi_i(h) = [u^i, h]$. Then a necessary condition that h maximize $f'(x,h)$ subject to $\phi(h) = \phi_i(h) = 0$ is that there exists c_i, $(1 \leq i \leq m+1)$, such that $f'(x)k =$

$$c_1 p \int_0^1 |h(t)|^{p-2}(h_1(t)k_1(t) + h_2(t)k_2(t))dt + \sum_{j=2}^{m+1} c_j [u^{j-1}, k] \quad \text{for all } k \in \mathcal{L}_p.$$

It follows that

$$p|x(t)|^{p-2}x_i(t) = pc_1|h(t)|^{p-2}h_i(t) + c_2 u_i^1(t) + \ldots + c_{m+1}u_i^m(t) \quad i=1,2.$$

Let $f_i(t) = (pc_1)^{-1}[p|x(t)|^{p-2}x_i(t) - c_2 u_i^1(t) - \ldots - c_{m+1}u_i^m(t)]$, and observe that

$$|h(t)|^{2p-2} = f_1^2(t) + f_2^2(t).$$

Therefore:

$$h_i(t) = [(f_1^2(t) + f_2^2(t)^{\frac{1}{2}}]^{\frac{q}{p}} f_i(t)/(f_1^2(t) + f_2^2(t))^{\frac{1}{2}}$$

showing that $h_i \in L_p$. We now solve the non-linear equations $\phi(h) = 0$, $\phi_i(h) = 0$ $(1 \leq i \leq m)$ for $c_2,\ldots,c_{m+1}$. If necessary, we replace h by $-h$ to ensure that we have a maximizer rather than a minimizer. The solution is now unique, due to the strict convexity of the sphere in $\mathcal{L}_p$. Because of the uniqueness of the extremal, the h we have constructed must be this extremal.

The subspace N is also an $\mathcal{L}_p$ space. Minimizing $f(x)$ on M is equivalent to minimizing $f(y + x_0)$ on N, with $x = y + x_0$. Clearly the gradient of the function f restricted to N is $h[f'(x),h]$. (See I2, above.) It follows, therefore, if $\phi(x) = h[f'(x),h]$, then ϕ satisfies the conditions required for the theorem of I.

By the theorem of I we may infer that every weak cluster point z of the sequence $\{x^k\}$ minimizes f and $f(x^k)$ converges downward to $f(z)$. Furthermore, since z is unique in the above problem, $\{x^k\} \rightharpoonup z$. We now show, moreover, that $\{x^k\} \to z$. For each component x^i of x, $(1 \leq i \leq n)$ we have that $0 \leq \int_0^1 |x^i(t)|^p dt \leq \int_0^1 |x(t)|^p dt$. Since $\{f(x_k)\}$ converges, the numbers $y_k^i = \int_0^1 |x_k^i(t)|^p dt$ are bounded and the sequence $\{x_k^i\}$ has a weak cluster point. In fact, $\{y_k^i\}$ converges. To prove this, observe that since $\{x_k\} \rightharpoonup z$, $\{x_k^i\} \rightharpoonup z_i$. Take a subsequence $\{x_k^i\}$ such that $\{y_k^i\} \to y^i$. Since $\{x_k^i\}$ converges both weakly and in L_p norm, $\{x_k^i\}$ converges strongly. Thus $\{x_k\} \to z'$, say. By continuity $f(x_k) \to f(z') = f(z)$. By uniqueness, $z = z'$. Therefore, the subsequence $\{x_k^i\} \to z^i$. Suppose that $\{y_k^i\}$ had another cluster point $\bar{y}^i \neq y^i$.

Take a new subsequence $\{x_k^i\}$ such that $\{v_k^i\} \to \bar{v}^i$. Again $\{x_k^i\} \to z^i$, and therefore, $\{v_k^i\} \to \int_0^T |z^i(\cdot)|^p dt = v^i$, a contradiction. It follows, therefore, that $\{x_k\} \to z$.

(b) The above processes require that at each cycle a non-linear system be solved to determine the gradient. This can be circumvented by imbedding the problem into a Hilbert space. Specifically, assume that the components of u^i are bounded and measurable. Let $\mathcal{L}_2$ denote the direct sum of $L_2[0,1]$ analogously to the above, and define $M' = \{x \in \mathcal{L}_2 : [u^i,x] = c^i : 1 \le i \le m\}$. Let f now be defined on M'. Since f achieves a minimum on M and $M \subset M'$, f also achieves a minimum on M'. Because M is dense in M', the minima are equal. The gradient of f on M' is merely the restriction of the gradient of f in $\mathcal{L}_2$ to M' and is obtained by orthogonal projection. See [9]. In general, $f'(x)$ does not exist. But if x is bounded and measurable, i.e., $x \in \mathcal{L}_\infty$, $f'(x) \in \mathcal{L}_2^*$ and $\nabla f(x) \in \mathcal{L}_2$. The set S is bounded in $\mathcal{L}_p$ and this implies S is bounded in $\mathcal{L}_2$, since $||x||_2 \le ||x||_p$ if $p \ge 2$.

Since f is convex and continuous, S is closed bounded and convex; furthermore, the derivatives of f are densely defined on S. Assume $x_n \in M'$, $x_n \in \mathcal{L}_\infty$ and $u^i \in \mathcal{L}_\infty$, $(1 \le i \le m)$. Then x_{n+1} is well defined and is also in $\mathcal{L}_\infty$. To see this, verify that $\nabla f(x_n) \in \mathcal{L}_\infty$ and the projection of $\nabla f(x_n)$ on the set $\{x \in \mathcal{L}_2 : [u^i,x] = 0, (1 \le i \le n)\}$ is also in $\mathcal{L}_\infty$. We are able to conclude again therefore that there is a unique minimizer z, and that $\{x_k\} \to z$ and $f(x_k) \searrow f(z)$.

REFERENCES

[1] Goldstein, A. A. "On steepest descent", Boeing document D1-82-0413, Seattle, Washington, February, 1965; *SIAM Control Series A*, Vol. 3, No. 1.

[2] Goldstein, A. A. "On Newton's method", Boeing document D1-82-0426, Seattle, Washington, April, 1965.

[3] Vainberg, M. M. "On the convergence of the method of steepest descents for non-linear equations", *Amer. Math. Soc. Doklady Translations* 1, 1 (1960).

[4] Neustadt, L. W. "Optimization, a moment problem and nonlinear programming", *Journ. SIAM Control*, Vol. 2, No. 1, 1964.

[5] Vainberg, M. M. Variational Methods for the Study of Non-linear Operators, Holden-Day, Inc., 1964.

[6] Riesz, F. and Sz-Nagy, B. Lecons d'Analyse Fonctionelle.

[7] Kantorovich, L. V. and Akilov, G. P. Functional Analysis in Normed Spaces, Macmillan, 1964.

[8] Goldstein, A. A. "Fuel optimization in orbital rendezvous", *Proceedings of the AIAA Conference on Guidance and Control*, Cambridge, Mass., August, 1963, with A. Green, A. Johnson, in Vol. 13 *Progress in Astronautics*, Academic Press, April, 1964, pp. 823-844.

[9] Goldstein, A. A. "Minimizing functionals on Hilbert space", appeared Computing Methods in Optimization Problems, Academic Press, April, 1964, pp. 159-166.

[10] Lusternik, L. A. "On conditional extrema of functionals", *Mat. Sb.* 41, 3, (1934), pp. 390-401. See also L. A. Lusternik and V. J. Sobolev, Elements of Functional Analysis, Ungar N. Y., 1961, p. 210.

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